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POSITIVE FRACTIONAL AND CONE FRACTIONAL LINEAR SYSTEMS WITH DELAYS

Tadeusz Kaczorek

Bialystok University of Technology, Poland

kaczorek@isep.pw.edu.pl

Abstract: The positive and cone fractional continuous-time and discrete-time linear systems are addressed. Sufficient conditions for the reachability of positive and cone fractional continuous-time linear systems are given. Necessary and sufficient conditions for the positivity and asymptotic stability of the continuous-time linear systems with delays are established. The realization problem for positive fractional continuous-time systems is formulated and solved. Necessary and sufficient conditions for the positivity and practical stability of fractional linear discrete-time systems are established. The linear matrix inequality (LMI) approaches are applied to testing the asymptotic stability of the positive fractional discrete-time linear systems. Sufficient conditions for the existence are established and procedures for computation of positive and cone realizations of the discrete-time linear systems are proposed.

Key words: fractional, positive, continuous-time, discrete-time, linear, system, reachability, controllability, realization problem, LMI approach.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems is given in the monographs [8, 13]. The stability and robust stability of positive and fractional 1D linear systems has been investigated in many papers and books [1-8, 11-14] as well as of 2D linear systems [15, 16, 21, 36, 39]. The realization problem of positive continuous-time and discrete-time linear systems has been considered in [18, 20, 22, 24, 29, 31, 37]. Recently, the reachability, controllability and minimum energy control of positive linear discrete-time systems with time delays have been considered in [47].

The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19th century [50-52, 54, 55]. This idea was used by engineers for modeling different processes in the late 1960s. The mathematical fundamentals of fractional calculus are given in monographs [51, 52, 54, 55]. The fractional order controllers were developed in [54]. Some other applications of fractional order systems can be found in [53, 60, 61].

The main purpose of this paper is to give an overview of some recent results on positive fractional and cone fractional continuous-time and discrete-time linear systems with delays.

The paper is organized as follows. In section 2 the positive and cone fractional linear continuous-time systems are introduced and sufficient conditions for the reachability are established. Necessary and sufficient conditions for the positivity and asymptotic stability of continuous-time system with delays are given in section 3. The fractional discrete-time systems and their practical stability are addressed in section 4. The LMI approaches to testing the asymptotic stability of the fractional systems are applied in section 5. The cone realization problem for discrete-time linear systems is formulated and solved in section 6. Concluding remarks and open problems are outlined in section 7.

The following notation will be used: $\Re$ - the set of real numbers; $\Re^{n\times m}$ - the set of $n\times m$ real matrices; $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries; $\Re_+^n = \Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries); I_n - the $n\times n$ identity matrix.

2. Pasitive fractional continuous-time linear systems and cone fractional systems

The following Caputo definition of the fractional derivative will be used as follows [27, 44, 52]

$$\frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(k-\alpha)} \int_0^t \frac{f^{(k)}(\tau)}{(t-\tau)^{\alpha+1-k}} d\tau, \qquad (1)$$

$k-1<\alpha\le k\in N=\{1,2,...\}$, where $\alpha\in\Re$ is the order of fractional derivative and $f^{(n)}(\tau)=\frac{d^k f(\tau)}{d\tau^k}$.

Consider the continuous-time fractional linear system described by the state equations

$$\frac{d^\alpha}{dt^\alpha}x(t)=Ax(t)+Bu(t),\ \ 0<\alpha\le 1, \tag{2a}$$

$$y(t)=Cx(t)+Du(t), \tag{2b}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$.

Theorem 1. [44] The solution of equation (2a) is given by

$$x(t)=\Phi_0(t)x_0+\int_0^t\Phi(t-\tau)Bu(\tau)d\tau,\ \ x(0)=x_0 \tag{3}$$

where

$$\Phi_0(t)=E_\alpha(At^\alpha)=\sum_{k=0}^{\infty}\frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \tag{4}$$

$$\Phi(t)=\sum_{k=0}^{\infty}\frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{5}$$

and $E_\alpha(At^\alpha)$ is the Mittag-Leffler matrix function, $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$ is the gamma function.

Definition 1. [44] The system (2) is called the internally positive fractional system if and only if $x(t)\in\Re_+^n$ and $y(t)\in\Re_+^p$ for $t\ge 0$ for any initial conditions $x_0\in\Re_+^n$ and all inputs $u(t)\in\Re_+^m$, $t\ge 0$.

Theorem 2. [44] The continuous-time fractional system (2) is internally positive if and only if the matrix A is a Metzler matrix and

$$A\in M_n,\ B\in\Re_+^{n\times m},\ C\in\Re_+^{p\times n},\ D\in\Re_+^{p\times m}. \tag{6}$$

Following [18, 26] the definitions are recalled.

Definition 2. Let $P=\begin{bmatrix}p_1\\ \vdots\\ p_n\end{bmatrix}\in\Re^{n\times n}$ be nonsingular and p_k be the k-th ($k=1,2,\dots,n$) its row.

The set

$$\mathcal{P}:=\left\{x\in\Re^n:\bigcap_{k=1}^{n}p_k x\ge 0\right\} \tag{7}$$

is called the linear cone generated by the matrix P.

In a similar way we may define for the inputs u the linear cone

$$\mathcal{Q}:=\left\{u\in\Re^m:\bigcap_{k=1}^{m}q_k u\ge 0\right\} \tag{8}$$

generated by the nonsingular matrix $Q=\begin{bmatrix}q_1\\ \vdots\\ q_m\end{bmatrix}\in\Re^{m\times m}$ and for the outputs y the linear cone

$$\mathcal{V}:=\left\{y\in\Re^p:\bigcap_{k=1}^{p}v_k y\ge 0\right\} \tag{9}$$

generated by the nonsingular matrix $V=\begin{bmatrix}v_1\\ \vdots\\ v_p\end{bmatrix}\in\Re^{p\times p}$.

Definition 3. The fractional system (2) is called ($\mathcal{P},\mathcal{Q},\mathcal{V}$) cone fractional system if $x(t)\in\mathcal{P}$ and $y(t)\in\mathcal{V}$, $t\ge 0$ for every $x_0\in\mathcal{P}$, $u(t)\in\mathcal{Q}$, $t\ge 0$.

The ($\mathcal{P},\mathcal{Q},\mathcal{V}$) cone fractional system (2) will be shortly called the cone fractional system. Note that if $\mathcal{P}=\Re_+^n$, $\mathcal{Q}=\Re_+^m$, $\mathcal{V}=\Re_+^n$ then the $(\Re_+^n,\Re_+^m,\Re_+^p)$ cone system is equivalent to the classical positive system [18, 26].

Theorem 3. The fractional system (2) is ($\mathcal{P},\mathcal{Q},\mathcal{V}$) cone fractional system if and only if

$$\bar{A}=PAP^{-1}\in\Re_+^{n\times n},\ \bar{B}=PBQ^{-1}\in\Re_+^{n\times m},$$
$$\bar{C}=VCP^{-1}\in\Re_+^{p\times n},\ \bar{D}=VDQ^{-1}\in\Re_+^{p\times m}. \tag{10}$$

Proof is given in [34].

3. Positive continuous-time systems with delays and their asymptotic stability

Consider the continuous-time linear system with q delays in state

$$\dot{x}(t)=A_0x(t)+\sum_{k=1}^{q}A_kx(t-d_k)+Bu(t) \tag{11a}$$

$$y(t)=Cx(t)+Du(t) \tag{11b}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors, A_k, $k=0,1,...,q$; B, C, D are real matrices of appropriate dimensions and d_k, $k=1,2,...,q$ are delays $(d_k\ge 0)$.

The initial conditions for (11a) have the form

$$x(t)=x_0(t)\ \text{for}\ t\in[-d,0],\ \ d=\max_k d_k \tag{12}$$

where $x_0(t)$ is a given vector function.

Definition 4. The system (11) is called (internally) positive if and only if $x(t) \in \Re_+^n$, $y \in \Re_+^n$ for any $x_0(t) \in \Re_+^n$ and for all inputs $u(t) \in \Re_+^m$, $t \geq 0$.

Theorem 4. The system (11) is (internally) positive if and only if

$$A_0 \in M_n, \quad A_k \in \Re_+^{n \times n}, \quad k = 1,...,q, \quad B \in \Re_+^{n \times m}, \quad C \in \Re_+^{p \times n}, \quad D \in \Re_+^{p \times m} \tag{13}$$

Proof is given in [35].

The positive system (11) is called asymptotically stable if and only if the solution of (11a) for $u(t) = 0 \in \Re_+^m$ satisfies the condition $\lim_{t \to \infty} x(t) = 0$ for $x_0(t) \in \Re_+^n$, $t \in [-d, 0]$.

Definition 5. Let a constant input $u(t) = u \in \Re_+^m$ be applied to the positive asymptotically stable system (11). A vector $x_e \in \Re_+^n$ satisfying the equality

$$0 = \sum_{k=0}^{q} A_k x_e + Bu \tag{14}$$

is called the equilibrium point of the system (11) corresponding to the input *u*.

If the positive system (11) is asymptotically stable, then the matrix

$$A = \sum_{k=0}^{q} A_k \in M_n \tag{15}$$

is nonsingular, and from (14) we have

$$x_e = -A^{-1} Bu \tag{16}$$

Theorem 5. The equilibrium point x_e corresponding to strictly positive $Bu > 0$ of the positive asymptotically stable system (11) is strictly positive, i.e. $x_e > 0$.

Remark 1. For the positive asymptotically stable system (11)

$$-A^{-1} \in \Re_+^{n \times n} \tag{17}$$

This follows immediately from (16) since $Bu \in \Re_+^n$ is arbitrary.

Theorem 6. The positive system (11) is asymptotically stable if and only if a strictly positive vector $\lambda \in \Re_+^n$ exists and satisfies the equality

$$A\lambda < 0, \quad A = \sum_{k=0}^{q} A_k \tag{18}$$

Proof is given in [35].

Remark 2. As a strictly positive vector λ the equilibrium point (16) of the system can be chosen, since

$$A\lambda = A(-A^{-1}Bu) = -Bu < 0 \quad \text{for} \quad Bu > 0 \tag{19}$$

Theorem 7. The positive system with delays (11) is asymptotically stable if and only if the positive system without delays

$$\dot{x} = Ax, \quad A = \sum_{k=0}^{q} A_k \in M_n \tag{20}$$

is asymptotically stable.

Proof is given in [35].

From Theorem 7 it follows that the checking of the asymptotic stability of positive systems with delays (11) can be reduced to checking the asymptotic stability of corresponding positive systems without delays (20). To check the asymptotic stability of positive system (11) the following theorem can be used.

Theorem 8. [44, 45] The positive system with delays (11) is asymptotically stable if and only if one of the following equivalent conditions holds:

1) Eigenvalues $s_1, s_2, ..., s_n$ of the matrix *A* have negative real parts, $\mathrm{Re}\, s_k < 0, k = 1,...,n$

2) All coefficients of the characteristic polynomial of the matrix *A* are positive

3) All leading principal minors of the matrix

$$-A = \begin{bmatrix} a_{11} & ... & a_{1n} \\ \vdots & ... & \vdots \\ a_{n1} & ... & a_{nn} \end{bmatrix} \tag{21}$$

are positive, i.e.

$$|a_{aa}| > 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \ldots, \det[-A] > 0 \tag{22}$$

Example 1. Using the conditions 2) and 3) of Theorem 8, let us check the asymptotic stability of the positive system (11) for *q* = 1 with the matrices

$$A_0 = \begin{bmatrix} -1 & 0.3 \\ 0.2 & -1.4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \tag{23}$$

The characteristic polynomial of the matrix

$$A = A_0 + A_1 = \begin{bmatrix} -0.5 & 0.4 \\ 0.4 & -0.6 \end{bmatrix} \tag{24}$$

has the form

$$\det[I_n s - A] = \begin{vmatrix} s + 0.5 & -0.4 \\ -0.4 & s + 0.6 \end{vmatrix} = s^2 + 1.1s + 0.14 \tag{25}$$

and its coefficients are positive.

Leading principal minors of the matrix

$$-A = \begin{bmatrix} 0.5 & -0.4 \\ -0.4 & 0.6 \end{bmatrix} \tag{26}$$

are positive, since $\Delta_1 = 0.5$, $\det[-A] = 0.14$.

Therefore, the conditions 2) and 3) of Theorem 8 are satisfied and the positive system (11) with (23) is asymptotically stable.

Theorem 9. The positive system with delays (11) is unstable for any matrices A_k, $k = 1,\ldots,q$ if the positive system

$$\dot{x} = A_0 x \tag{27}$$

is unstable.

Proof. By Theorem 6 if the system (27) is unstable, then a strictly positive vector $\lambda \in \Re_+^n$ does not exist as far as $A_0\lambda < 0$. In this case a strictly positive vector $\lambda \in \Re_+^n$ satisfying the inequality (18) does not exist, since for the positive system $A_k \in \Re_+^{n\times n}$ and $A_k\lambda \in \Re_+^n$, $k = 1,\ldots,q$.

Theorem 10. [35] If at least one diagonal entry of the matrix A_0 is positive, the positive system (11) is unstable for any $A_k \in \Re_+^{n\times n}$, $k = 1,\ldots,q$.

These considerations can be extended to positive fractional continuous-time systems with delays.

4. Positive fractional discrete-time systems and their practical stability

In this paper the following definition of the fractional discrete derivative

$$\Delta^\alpha x_k = \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x_{k-j}, \quad 0 < \alpha < 1 \tag{28}$$

is used, where $\alpha \in \Re$ is the order of the fractional discrete difference, and

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j = 0 \\ \dfrac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} & \text{for } j = 1,2,\ldots \end{cases} \tag{29}$$

Consider the fractional discrete linear system, described by the state-space equations

$$\Delta^\alpha x_{k+1} = Ax_k + Bu_k, \quad k \in Z_+ \tag{30a}$$

$$y_k = Cx_k + Du_k \tag{30b}$$

where $x_k \in \Re^n$, $u_k \in \Re^m$, $y_k \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

Using the definition (28), we may write the equations (30) in the form

$$x_{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} = Ax_k + Bu_k, \quad k \in Z_+ \tag{31a}$$

$$y_k = Cx_k + Du_k \tag{31b}$$

Definition 6. The system (31) is called the (internally) positive fractional system if and only if $x_k \in \Re_+^n$ and $y_k \in \Re_+^p$, $k \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$ and all input sequences $u_k \in \Re_+^m$, $k \in Z_+$

Theorem 11. The solution of equation (31a) is given by

$$x_k = \Phi_k x_0 + \sum_{i=0}^{k-1} \Phi_{k-i-1} Bu_i \tag{32}$$

where Φ_k is determined by the equation

$$\Phi_{k+1} = (A + I_n\alpha)\Phi_k + \sum_{i=2}^{k+1} (-1)^{i+1} \binom{\alpha}{i} \Phi_{k-i+1} \tag{33}$$

with $\Phi_0 = I_n$.

The proof is given in [25, 44].

Lemma 1. According to [25, 44] if

$$0 < \alpha \le 1 \tag{34}$$

then

$$(-1)^{i+1} \binom{\alpha}{i} > 0 \quad \text{for } i = 1,2,\ldots \tag{35}$$

Theorem 12. According to [25, 44] let $0 < \alpha < 1$. Then the fractional system (31) is positive if and only if

$$A + I_n\alpha \in \Re_+^{n\times n}, \; B \in \Re_+^{n\times m}, \; C \in \Re_+^{p\times n}, \; D \in \Re_+^{p\times m} \tag{36}$$

From (29) and (35) it follows that the coefficients

$$c_j = c_j(\alpha) = (-1)^j \binom{\alpha}{j+1}, \quad j = 1,2,\ldots \tag{37}$$

decrease steeply with increasing j and they are positive for $0 < \alpha < 1$. In practical problems it is assumed that j is bounded by some natural number h. In this case the equation (31a) takes the form

$$x_{k+1} = A_\alpha x_k + \sum_{j=1}^{h} c_j x_{k-j} + Bu_k, \quad k \in Z_+ \tag{38}$$

where

$$A_\alpha = A + I_n\alpha \tag{39}$$

Note that the equations (38) and (31b) describe a linear discrete-time system with h delays in state.

Definition 7. The positive fractional system (31) is called practically stable if and only if the system (38), (31b) is asymptotically stable.

Defining the new state vector

$$\tilde{x}_k = \begin{bmatrix} x_k \\ x_{k-1} \\ \vdots \\ x_{k-h} \end{bmatrix} \tag{40}$$

we may write the equations (38) and (31b) in the form

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}u_k, \quad k \in Z_+ \tag{41a}$$

$$y_k = \tilde{C}x_k + \tilde{D}u_k \tag{41b}$$

where

$$\tilde{A} = \begin{bmatrix} A_\alpha & c_1 I_n & c_2 I_n & \dots & c_{h-1} I_n & c_h I_n \\ I_n & 0 & 0 & \dots & 0 & 0 \\ 0 & I_n & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I_n & 0 \end{bmatrix} \in \Re_+^{\tilde{n}\times\tilde{n}}, \ \tilde{B} = \begin{bmatrix} B \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \Re_+^{\tilde{n}\times m}$$

$$\tilde{C} = [C \ \ 0 \ \ \dots \ \ 0] \in \Re_+^{p\times\tilde{n}}, \ \tilde{D} = D = \in \Re_+^{p\times m}, \quad \tilde{n} = (1+h)n \tag{41c}$$

To test the practical stability of the positive fractional system (31) the well-known conditions for positive systems [13] can be applied to the system (41).

Theorem 13. The positive fractional system (38) is practically stable if and only if one of the following conditions is satisfied:

1) the moduli of eigenvalues $\tilde{z}_k$, $k = 1,\dots,\tilde{n}$ of the matrix $\tilde{A}$ are less then 1, i.e.

$$|\tilde{z}_k| < 1 \text{ for } k = 1,\dots,\tilde{n}, \tag{42}$$

2) $\det[zI_{\tilde{n}} - \tilde{A}] \neq 0$ for $|z| < 1$,

3) $\rho(\tilde{A}) < 1$, where $\rho(\tilde{A})$ is the spectral radius defined by $\rho(\tilde{A}) = \max_{1\le k\le\tilde{n}}\{|\tilde{z}_k|\}$ of the matrix $\tilde{A}$,

4) all coefficients $\tilde{a}_i$, $i = 0,1,\dots,\tilde{n}-1$ of the characteristic polynomial

$$p_{\tilde{A}}(z) = \det[I_{\tilde{n}}(z+1) - \tilde{A}] = z^{\tilde{n}} + \tilde{a}_{\tilde{n}-1}z^{\tilde{n}-1} + \dots + \tilde{a}_1 z + \tilde{a}_0 \tag{43}$$

of the matrix $[\tilde{A} - I_{\tilde{n}}]$ are positive.

All principal minors of the matrix

$$(I_{\tilde{n}} - \tilde{A}) = \begin{bmatrix} \tilde{a}_{11} & \dots & \tilde{a}_{1\tilde{n}} \\ \vdots & \dots & \vdots \\ \tilde{a}_{\tilde{n}1} & \dots & \tilde{a}_{\tilde{n}\tilde{n}} \end{bmatrix} \tag{44}$$

are positive , i.e.

$$|\tilde{a}_{11}| > 0, \quad \begin{vmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{vmatrix} > 0, \dots, \det[I_{\tilde{n}} - \tilde{A}] > 0 \tag{45}$$

There exist strictly positive vectors $\bar{x}_i \in \Re_+^n$, $i = 0,1,\dots,h$ satisfying

$$\bar{x}_0 < \bar{x}_1, \ \bar{x}_1 < \bar{x}_2, \dots, \bar{x}_{h-1} < \bar{x}_h \tag{46}$$

such that

$$A_\alpha \bar{x}_0 + c_1\bar{x}_1 + \dots + c_h\bar{x}_h < \bar{x}_0 \tag{47}$$

Proof is given in [17, 44].

Example 2. Check the practical stability of the positive fractional system

$$\Delta^\alpha x_{k+1} = 0.1 x_k, \quad k \in Z_+ \tag{48}$$

for $\alpha = 0.5$ and $h = 2$.

Using (37), (39) and (41c), we obtain

$$c_1 = \frac{\alpha(1-\alpha)}{2} = \frac{1}{8}, \ c_2 = \frac{1}{16}, \ a_\alpha = 0.6 \tag{49}$$

and

$$\tilde{A} = \begin{bmatrix} a_\alpha & c_1 & c_2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0.6 & \frac{1}{8} & \frac{1}{16} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{50}$$

In this case the characteristic polynomial (43) has the form

$$p_{\tilde{A}}(z) = \det[I_{\tilde{n}}(z+1) - \tilde{A}] = \begin{vmatrix} z+0.4 & -0.125 & -0.0625 \\ -1 & z+1 & 0 \\ 0 & -1 & z+1 \end{vmatrix} = \tag{51}$$

$$= z^3 + 2.4z^2 + 1.675z + 0.2125$$

All coefficients of the polynomial (51) are positive and by Theorem 13 the system is practically stable.

Theorem 14. The positive fractional system (31) is practically stable only if the positive system

$$x_{k+1} = A_\alpha x_k, \quad k \in Z_+ \tag{52}$$

is asymptotically stable.

Proof. From (47) we have

$$(A_\alpha - I_n)\bar{x}_0 + c_1\bar{x}_1 + \dots + c_h\bar{x}_h < 0. \tag{53}$$

Note that the inequality (53) may be satisfied only if such a strictly positive vector $\bar{x}_0 \in \Re_+^n$ exists that

$$(A_\alpha - I_n)\bar{x}_0 < 0 \tag{54}$$

and since $c_1\bar{x}_1 + \dots + c_h\bar{x}_h > 0$.

The condition (54) implies the asymptotic stability of the positive system (52).

From Theorem 14 we have the following important corollary.

Corollary 1. The positive fractional system (31) is practically unstable for any finite *h* if the positive system (52) is unstable.

Theorem 15. The positive fractional system (31) is practically unstable if at least one diagonal entry of the matrix A_α is greater than 1.

Proof. The proof follows immediately from Theorems 14 and 13.

Example 3. Consider the autonomous positive fractional system described by the equation

$$\Delta^\alpha x_{k+1} = \begin{bmatrix} -0.5 & 1 \\ 2 & 0.5 \end{bmatrix} x_k, \quad k \in Z_+ \tag{55}$$

for $\alpha = 0.8$ and any finite *h*. In this case $n = 2$ and

$$A_\alpha = A + I_n\alpha = \begin{bmatrix} 0.3 & 1 \\ 2 & 1.3 \end{bmatrix} \tag{56}$$

By Theorem 15 the positive fractional system is practically unstable for any finite h since the entry (2,2) of the matrix (56) is greater than 1. The same result follows from the fact that the characteristic polynomial of the matrix $A_\alpha - I_n$

$$p_A(z) = \det[I_{\tilde{n}}(z+1) - A] = \begin{bmatrix} z+0.7 & -1 \\ -2 & z+2.3 \end{bmatrix} = z^2 + 0.4z - 2.21 \tag{57}$$

has one negative coefficient $\hat{a}_0 = -2.21$.

5. Application of LMI approach to positive discrete-time systems

Definition 8. [38] An inequality of the form

$$F(x) + F > 0 \tag{58}$$

where x takes values in the real vector space V, the mapping $F : V \to S^n$ is linear, and $F \in S^n$, is called the linear matrix inequality (LMI). The LMI is called feasible if such $x \in V$ exists that the inequality (58) is satisfied; otherwise the LMI is called infeasible.

Lemma 2. [38] A nonnegative matrix $A = \Re_+^{n\times n}$ is Schur matrix if and only if the LMI

$$blockdiag\,[P - A^T PA, \quad P] \succ 0 \tag{59}$$

is feasible with respect to the diagonal matrix P.

Lemma 3. [38] A Metzler matrix $A = \Re^{n\times n}$ is Hurwitz matrix if and only if the LMI

$$blockdiag\,[-(A^T P + PA), \quad P] \succ 0 \tag{60}$$

is feasible with respect to the diagonal matrix P.

It is well-known [38] that $A = \Re_+^{n\times n}$ is Schur matrix if and only if $(A - I_n)$ is Hurwitz matrix.

Lemma 4. [38] A nonnegative matrix $A = \Re_+^{n\times n}$ is Schur matrix if and only if the LMI

$$blockdiag\,[-\left((A - I_n)^T P + P(A - I_n)\right), \quad P] \succ 0 \tag{61}$$

is feasible with respect to the diagonal matrix P.

Lemma 5. [38] A nonnegative matrix $A = \Re_+^{n\times n}$ is Schur matrix if and only if the LMI

$$blockdiag\left\{\begin{bmatrix} P & -A^T P \\ -PA & P \end{bmatrix}, \ P\right\} \succ 0 \tag{62}$$

is feasible with respect to the diagonal matrix P.

Theorem 16. The positive fractional system (31) is practically stable if and only if one of the following equivalent conditions

$$blockdiag\left\{\begin{bmatrix} P_1 - P_2 - A_\alpha^T P_1 A_\alpha & -c_1 A_\alpha^T P_1 & \dots & -c_{h-1}A_\alpha^T P_1 & -c_h A_\alpha^T P_1 \\ -c_1 P_1 A_\alpha & P_2 - P_3 - c_1^2 P_1 & \dots & -c_1 c_{h-1} P_1 & -c_1 c_h P_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_{h-1}P_1 A_\alpha & -c_1 c_{h-1} P_1 & \dots & P_h - P_{h+1} - c_{h-1}^2 P_1 & -c_{h-1}c_h P_1 \\ -c_h P_1 A_\alpha & -c_1 c_h P_1 & \dots & -c_{h-1}c_h P_1 & P_{h+1} - c_h^2 P_1 \end{bmatrix}, \begin{bmatrix} P_1 & 0 & \dots & 0 & 0 \\ 0 & P_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_h & 0 \\ 0 & 0 & \dots & 0 & P_{h+1} \end{bmatrix}\right\} \succ 0 \tag{63}$$

is feasible with respect to the diagonal matrices $P_1, \dots, P_{h+1}$.

2) The LMI

$$blockdiag\left\{\begin{bmatrix} A_\alpha^T P_1 + P_1 A_\alpha - 2P_1 & P_2 + c_1 P_1 & \dots & c_{h-1}P_1 & c_h P_1 \\ P_2 + c_1 P_1 & -2P_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{h-1}P_1 & 0 & \dots & -2P_{h-1} & P_{h+1} \\ c_h P_1 & 0 & \dots & P_{h+1} & -2P_h \end{bmatrix}, \begin{bmatrix} P_1 & 0 & \dots & 0 & 0 \\ 0 & P_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_h & 0 \\ 0 & 0 & \dots & 0 & P_{h+1} \end{bmatrix}\right\} \succ 0 \tag{64}$$

is feasible with respect to the diagonal matrices $P_1, \dots, P_{h+1}$.

3) The LMI

$$blockdiag\left\{\begin{bmatrix} P_1 & 0 & \dots & 0 & -A_\alpha^T P_1 & -P_2 & \dots & 0 \\ 0 & P_2 & \dots & 0 & -c_1 P_1 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & P_{h+1} & -c_h P_1 & 0 & \dots & -P_{h+1} \\ -P_1 A_\alpha & -c_1 P_1 & \dots & -c_h P_1 & P_1 & 0 & \dots & 0 \\ -P_2 & 0 & \dots & 0 & 0 & P_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & -P_{h+1} & 0 & 0 & \dots & P_{h+1} \end{bmatrix}, \begin{bmatrix} P_1 & 0 & \dots & 0 & 0 \\ 0 & P_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_h & 0 \\ 0 & 0 & \dots & 0 & P_{h+1} \end{bmatrix}\right\} \succ 0 \tag{65}$$

Proof is given in [38].

6. Cone-realization problem for discrete-time systems with delays

Consider the discrete-time linear systems with delays

$$x_{i+1} = A_0 x_i + A_1 x_{i-1} + B_0 u_1 + B_1 u_{i-1} \quad (66a)$$

$$y_i = Cx_i + Du_i,\ i \in Z_+ = \{0,1,...\} \quad (66b)$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $A_0, A_1 \in \Re^{n\times n}$, $B_0, B_1 \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

Using Definition 2, it is easy to show that [32] the transfer matrix

$$T(z) = C[I_n z^2 - A_0 z - A_1]^{-1}(B_0 z + B_1) + D \quad (67)$$

of the ($\mathcal{P},\mathcal{Q},\mathcal{V}$)-system (66) and the transfer matrix

$$\bar{T}(z) = \bar{C}[I_n z^2 - \bar{A}_0 z - \bar{A}_1]^{-1}(\bar{B}_0 z + \bar{B}_1) + \bar{D} \quad (68)$$

of the positive system (66) are related by the equality

$$\bar{T}(z) = VT(z)Q^{-1} \quad (69)$$

Consider the linear system (66) with its transfer matrix (67). Let $\Re^{p\times m}(z)$ be the set of $p\times m$ rational proper matrices.

Definition 9. Matrices $A_k \in \Re^{n\times n}$, $B_k \in \Re^{n\times m}$, $k = 0,1$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$ are called a ($\mathcal{P},\mathcal{Q},\mathcal{V}$)-cone realization of a given proper $T(z)$ if they satisfy the equality (67) and the conditions

$$PA_kP^{-1} \in \Re_+^{n\times n},\ PB_kQ^{-1} \in \Re_+^{n\times m},\ k = 0,1,$$
$$VCP^{-1} \in \Re_+^{p\times n},\ VDQ^{-1} \in \Re_+^{p\times m} \quad (70)$$

where *P*, *Q* and *V* are nonsingular matrices generating the cones $\mathcal{P},\mathcal{Q}$ and $\mathcal{V}$, respectively.

The ($\mathcal{P},\mathcal{Q},\mathcal{V}$)-cone realization problem can be stated as follows: being given a proper rational matrix $T(z) \in \Re^{p\times m}(z)$ and non-singular matrices *P*, *Q*, *V* generating cones $\mathcal{P},\mathcal{Q}$ and $\mathcal{V}$, find a ($\mathcal{P},\mathcal{Q},\mathcal{V}$)-cone realization of $T(z)$.

A procedure for computation of a ($\mathcal{P},\mathcal{Q},\mathcal{V}$)-cone realization of $T(z)$ will be proposed and solvability conditions of the problem will be established.

From (68) we have

$$\bar{D} = \lim_{z\to\infty} \bar{T}(z) \quad (71)$$

since $\lim_{z\to\infty}[z^{-1}(I_n z^2 - \bar{A}_0 z - \bar{A}_1)]^{-1} = 0.$

The strictly proper part of $\bar{T}(z)$ is given by

$$\bar{T}_{sp}(z) = \bar{T}(z) - \bar{D}. \quad (72)$$

It is easy to show that if the matrices A_0 and A_1 have the following forms

$$A_0 = \begin{bmatrix} 0 & ... & 0 & a_1 \\ 0 & ... & 0 & a_3 \\ 0 & ... & 0 & a_5 \\ ... & ... & ... & ... \\ 0 & ... & 0 & a_{2n-1} \end{bmatrix} \in \Re^{n\times n},$$
$$A_1 = \begin{bmatrix} 0 & 0 & ... & 0 & a_0 \\ 1 & 0 & ... & 0 & a_2 \\ 0 & 1 & ... & 0 & a_4 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 1 & a_{2(n-1)} \end{bmatrix} \in \Re^{n\times n} \quad (73)$$

then

$$d(z) = \det[I_n z^2 - A_0 z - A_1] = z^{2n} - a_{2n-1}z^{2n-1} - ... - a_1 z - a_0 \quad (74)$$

and the *n*-th row of the adjoint matrix $\mathrm{Adj}[I_n z^2 - A_0 z - A_1]$ has the form

$$R_n(z) = [1 \quad z^2 \quad z^4 \quad ... \quad z^{2(n-1)}] \quad (75)$$

The strictly proper $\bar{T}_{sp}(z)$ can be always written in the form

$$\bar{T}_{sp}(z) = \begin{bmatrix} \frac{N_1(z)}{d_1(z)} \\ \vdots \\ \frac{N_p(z)}{d_p(z)} \end{bmatrix} \quad (76)$$

where

$$d_i(z) = z^{2q_i} - a_{i2q_1-1}z^{2q_i-1} - ... - a_{i1}z - a_{i0},\quad i = 1,...,p \quad (77)$$

is the least common denominator of the *i*-th row of $\bar{T}_{sp}(z)$ and

$$N_i(z) = [n_{i1}(z) \quad n_{i2}(z) \quad ... \quad n_{im}(z)],\quad i = 1,...,p$$
$$n_{ij}(z) = n_{ij}^{2q_i-1}z^{2q_i-1} + ... + n_{ij}^1 z + n_{ij}^0,\quad j = 1,...,m \quad (78)$$

To the polynomial (77) we associate the pair of the matrices

$$\bar{A}_{0i} = \begin{bmatrix} 0 & ... & 0 & a_{i1} \\ 0 & ... & 0 & a_{i3} \\ 0 & ... & 0 & a_{i5} \\ ... & ... & ... & ... \\ 0 & ... & 0 & a_{i2q_i-1} \end{bmatrix} \in \Re^{q_i\times q_i},$$
$$\bar{A}_{1i} = \begin{bmatrix} 0 & 0 & ... & 0 & a_{i0} \\ 1 & 0 & ... & 0 & a_{i2} \\ 0 & 1 & ... & 0 & a_{i4} \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 1 & a_{i2(q_i-1)} \end{bmatrix} \in \Re^{q_i\times q_i}\ i = 1,...,p \quad (79)$$

satisfying the condition

$$d_i(z) = \det[I_{q_i} z^2 - \overline{A}_{0i} z - \overline{A}_{1i}], \quad i = 1,...,p \quad (80)$$

Let

$$\overline{C} = \text{block diag}[\overline{C}_1 \quad \overline{C}_2 \quad ... \quad \overline{C}_p],$$
$$\overline{C}_i = [0 \quad 0 \quad ... \quad 1] \in \Re^{1\times q_i}, \; i = 1,...,p \quad (81)$$

and

$$\begin{bmatrix} b_{1j}^k \\ b_{2j}^k \\ \vdots \\ b_{pj}^k \end{bmatrix}, \; b_{ij}^k = \begin{bmatrix} b_{ij}^{k1} \\ b_{ij}^{k2} \\ \vdots \\ b_{ij}^{kq_i} \end{bmatrix}, \; k = 0,1; i = 1,...,p; \; j = 1,...,m \quad (82)$$

be the j-th column of the matrix $\overline{B}_k$ $(k = 0,1)$.

It is easy to show [32] that the entries of B_k, $k = 0,1$ are given by

$$b_{1j}^{0q_1} = n_{1j}^{2q_1-1}, b_{1j}^{1q_1} = n_{1j}^{2(q_1-1)}, ..., b_{1j}^{01} = n_{1j}^1, b_{1j}^{11} = n_{1j}^0, \quad j = 1,...,m$$
$$...$$
$$b_{pj}^{0q_p} = n_{pj}^{2q_p-1}, b_{pj}^{1q_p} = n_{pj}^{2(q_p-1)}, ..., b_{pj}^{01} = n_{pj}^1, b_{pj}^{11} = n_{pj}^0 \quad (83)$$

Theorem 17. [32] There exists a positive realization $\overline{A}_k \in \Re_+^{n\times n}$, $\overline{B}_k \in \Re_+^{n\times m}$, $\overline{C} \in \Re_+^{p\times n}$, $\overline{D} \in \Re_+^{p\times m}$ of $\overline{T}(z)$ if

$$1) \; \overline{T}(\infty) = \lim_{z\to\infty}(\overline{T}(z)) \in \Re_+^{p\times m} \quad (84)$$

2) the coefficients of $d_i(z)$, $i = 1,...,p$ are nonnegative, i.e.

$$a_{ij} \geq 0 \;\text{ for } i = 1,...,p; \; j = 0,1,...,2q_i - 1 \quad (85)$$

3) $n_{ij}^k \geq 0$ for $i = 0,1,...,p; \; j = 1,...,m; \; k = 0,1,...,2q_i - 1$ (86)

If the conditions (84), (85) and (86) are satisfied, a positive realization of $\overline{T}(z)$ can be computed by the use of the following procedure.

Procedure 1.

Step 1. Using (71) and (72) find $\overline{D} \in \Re_+^{p\times m}$ and the strictly proper matrix $\overline{T}_{sp}(z)$.

Step 2. Knowing the coefficients a_{ij}, $(i = 0,1,...,p; \; j = 0,1,...,2q_i - 1)$ of $d_i(z), \; i = 1,...,p$, find the matrices (79) and

$$\overline{A}_0 = \text{block diag}[\overline{A}_{01},...,\overline{A}_{0p}] \in \Re_+^{n\times n},$$
$$\overline{A}_1 = \text{block diag}[\overline{A}_{11},...,\overline{A}_{1p}] \in \Re_+^{n\times n} \quad (87)$$

Step 3. Knowing the coefficients n_{ij}^k $(i = 0,1,...,p; \; j = 1,...,m; \; k = 0,1,...,2q_i - 1)$ of $N_i(z)$ $(i = 1,...,p)$ and using (82) find $\overline{B}_k$ for $k = 0,1$ and the matrix $\overline{C}$ of the form (81).

A $(\mathcal{P}, \mathcal{Q}, \mathcal{V})$-cone realization for a given $T(z) \in \Re^{p\times m}(z)$ and non-singular matrices P, Q, V can be computed by the use of the following procedure.

Procedure 2.

Step 1. Knowing $T(z)$ and the matrices V, Q and using (69) compute the transfer matrix $\overline{T}(z)$.

Step 2. Using Procedure 1, compute a positive realization $\overline{A}_k$, $\overline{B}_k$, $k = 0,1$, $\overline{C}$, $\overline{D}$ of the transfer matrix $\overline{T}(z)$.

Step 3. Using the relations

$$A_k = P^{-1}\overline{A}_k P, \; B_k = P^{-1}\overline{B}_k Q, \; k = 0,1,$$
$$C = V^{-1}\overline{C}P, \; D = V^{-1}\overline{D}Q \quad (88)$$

compute the desired realization.

Theorem 18. A $(\mathcal{P}, \mathcal{Q}, \mathcal{V})$-cone realization of $T(z)$ exists if and only if a positive realization of $\overline{T}(z)$ exists.

The proof follows immediately from Procedure 2

From Theorem 17 for single-input single-output system ($m = p = 1$) we obtain the following important corollary.

Corollary 2. There exists a $(\mathcal{P}, \mathcal{Q}, \mathcal{V})$-cone realization A_k, B_k, $k = 0,1$, C, D of the transfer function $T(z)$ if and only if there exist a positive realization $\overline{A}_k$, $\overline{B}_k$, $k = 0,1$, $\overline{C}$, $\overline{D}$ of $\overline{T}(z)$ and the realization are related by

$$A_k = P^{-1}\overline{A}_k P, \; B_k = P^{-1}\overline{B}_k g, \; k = 0,1, \; C = \overline{C}P$$
$$(\text{or } B_k = P^{-1}\overline{B}_k \text{ and } C = g\overline{C}P) \; D = k\overline{D} \quad (89)$$

where $g = QV^{-1}$ is a scalar.

For $m = p = 1$ the transfer functions $\overline{T}(z)$ and $T(z)$ relate to $\overline{T}(z) = gT(z)$.

7. Concluding remarks and open problems

The positive fractional and cone fractional linear continuous-time and discrete-time systems have been addressed. The cone fractional linear systems have been introduced. Necessary and sufficient conditions for the positivity and asymptotic stability of the continuous time linear systems with delays have been given. It has been shown that the checking of the asymptotic stability of these positive systems can be reduced to checking the asymptotic stability of corresponding positive systems without delays. Necessary and sufficient conditions for the positivity and practical stability of fractional linear discrete-time systems have been established. It has been shown that the LMI approaches can be successfully applied to testing the asymptotic stability of the positive fractional linear discrete-time systems. The realization problem for the discrete-time linear systems with delays have been formulated and solved. Sufficient conditions for the existence and procedures for computation of the positive and cone realizations have been proposed. Many of these results can be extended to 2D positive fractional linear systems.

Extensions of these considerations for the following classes of systems are open problems:

1) 1D and 2D varying positive linear systems,
2) 2D hybrid systems without and with delays,
3) 2D Lyapunov systems,
4) 1D and 2D positive fractional switching systems.

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ДОДАТНІ ДРОБОВІ ТА КОНІЧНІ ДРОБОВІ ЛІНІЙНІ СИСТЕМИ ІЗ ЗАТРИМКОЮ

Тадеуш Качорек

У статті розглянуто додатні та конічні дробові неперервні та дискретні в часі лінійні системи. Наведено достатні умови для досяжності таких систем. Встановлено необхідні та достатні умови для додатності та асимптотичної стабільності неперервних у часі лінійних систем із затримкою. Сформульовано та розв'язано проблему реалізації додатних дробових неперервних у часі систем. Встановлено необхідні та достатні умови для додатності та практичної стабільності дробових дискретних у часі лінійних систем Застосовано підхід лінійної матричної нерівності (ЛМН) для перевірки асимптотичної стабільності додатних дробових дискретних у часі лінійних систем. Встановлено достатні умови для існування та запропоновано процедури для розрахунку додатних та конічних реалізацій дискретних у часі лінійних систем.

Tadeusz Kaczorek – D.Sc., Professor, born in 1932 in Poland, received his M.Sc., Ph.D. and D.Sc. degrees in Electrical Engineering from Warsaw University of Technology, Poland, in 1956, 1962 and 1964, respectively. In the period 1968 – 1969 he was the dean of Electrical Engineering Faculty and in the period 1970 – 1973 – the pro-rector of Warsaw University of Technology, Poland. Since 1971 he has been a professor and since 1974 a full professor at Warsaw University of Technology, Poland. In 1986 he was elected a corresponding member and in 1996 full member of Polish Academy of Sciences. In the period 1988 - 1991 he was the director of the Research Centre of Polish Academy of Sciences in Rome. In June 1999 he was elected the full member of the Academy of Engineering in Poland.

In May 2004 he was elected to the honorary member of the Hungarian Academy of Sciences. He was awarded by the University of Zielona Gora, Poland (2002) by the title doctor honoris causa, the Technical University of Lublin, Poland (2004), the Technical University of Szczecin, Poland (2004), Warsaw University of Technology, Poland (2004), Bialystok University of Technology, Poland (2008), Lodz University of Technology, Poland (2009), Opole University of Technology, Poland (2009), Poznan University of Technology, Poland (2011), and Rzeszow University of Technology, Poland (2012).

His research interests cover the theory of systems and the automatic control systems theory, specially, singular multidimensional systems, positive multidimensional systems and singular positive 1D and 2D systems. He has initiated the research in the field of singular 2D, positive 2D linear systems and positive fractional 1D and 2D systems. He has published 25 books (7 in English) and over 1000 scientific papers.

He supervised 69 Ph.D. theses. More than 20 of these PhD students became professors in USA, UK and Japan. He is the editor-in-chief of the Bulletin of the Polish Academy of Sciences, Technical Sciences and the editorial member of about ten international journals.